

DENSITIES OF RATIONAL POINTS AND NUMBER FIELDS

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ABSTRACT. We relate the problem of counting number fields, in particular, Malle's conjecture with the problem of counting rational points on singular Fano varieties, in particular, Batyrev and Tschinkel's generalization of Manin's conjecture.

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1. INTRODUCTION

In this paper, we relate two subjects in the number theory: the density of rational points and the one of number fields.

As for the former, we consider an algebraic variety X over a number field K , and a suitable set $U \subset X(K)$ of K -points, obtained by removing accumulating subsets from $X(K)$. It is expected that the distribution of points in U reflects the geometry of X . We are interested in the case where X is a Fano variety. Then U tends to be an infinite set. Given a metric on the anti-canonical sheaf ω_X^{-1} , we can define a height function

$$H : X(K) \rightarrow \mathbb{R}_{>0}$$

so that for each $B \in \mathbb{R}_{>0}$,

$$N_U(B) := \#\{x \in U \mid H(x) \leq B\} < \infty.$$

We are interested in the asymptotic behavior of $N_U(B)$ as B tends to infinity. Manin's conjecture [FMT89] concerns this problem. A not so precise version states that

$$N_U(B) \sim c \cdot B \cdot (\log B)^{\rho-1},$$

with ρ the Picard number of X , under the assumption that X is smooth. Batyrev and Tschinkel [BT98] generalized it to Fano varieties having canonical singularities, where ρ was replaced with a number incorporating singularities.

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As for the density of number fields, let G be a transitive subgroup of the symmetric group S_n . We consider degree n extensions L/K of a given number field K such that its Galois closure has Galois group permutation isomorphic to G : we call such an extension L/K a G -field over K . We put $M_{G,K}(B)$ to be the number of G -fields with $|N_{K/\mathbb{Q}}(D_{L/K})| \leq B$, where $D_{L/K}$ is the discriminant of L/K and $N_{K/\mathbb{Q}}$ the norm of K/\mathbb{Q} . The asymptotic behavior of $M_{G,K}(B)$ as B tends to infinity is another concern of ours. Malle's conjecture [Mal02, Mal04] states that

$$M_{G,K}(B) \sim c \cdot B^{1/\text{ind}(G)} \cdot (\log B)^{\beta(G,K)-1}$$

for some constant c and invariants $\text{ind}(G)$, $\beta(G, K)$ determined by G and K .

We would like to relate Malle's conjecture, and Batyrev and Tschinkel's conjecture. Our strategy is to consider the quotient variety

$$X := (\mathbb{P}_K^m)^n / G$$

for the natural G -action on the power of the projective space. If $m \cdot \text{ind}(G) \geq 2$, then X is a Fano variety with only canonical singularities. We exhibit a correspondence between *primitive* K -points of X and *original* F -points of \mathbb{P}_K^m with F/K running over G -fields. Since the correspondence respect heights, we obtain an equality among height zeta functions of the form:

$$Z_{X(K)^{\text{prim}}}(s) = \frac{1}{\#Z(G)} \cdot \sum_{L \in G\text{-Fie}(K)} Z_{\mathbb{P}^m(L^{G_1})^{\text{orig}}}(s),$$

(for details, see Theorem 6.3). We expect that the left side contains information about the density of primitive K -points of X and the right side contains information about the density of G -fields. Using this equality together with additional conjectures, we show partial results on implications between Malle's conjecture, and Batyrev and Tschinkel's conjecture.

Finally we briefly mention relations to other works. In the paper [EV05, page 153] of Ellenberg and Venkatesh, it was mentioned, as a comment by Tschinkel, a similarity between their work on Malle's conjecture and Batyrev's one on rational points on Fano varieties. In another paper of theirs [EV06, page 732], the relation between Malle's conjecture and Manin's conjecture was more explicitly noted. In the same paper, they use the field of multi-symmetric functions, which is the function field of the above quotient variety X . The approach using X or its function field is regarded as a revisitation of Noether's approach to the inverse Galois problem (see [Ser08]), incorporating newer materials such as Malle's and Manin's conjectures. Our work is also a global analogue of the wild McKay correspondence [WY, Yas14, Yasa, Yasb], which relates weighted counts of extensions of a local field with stringy invariants of quotient varieties.

Throughout the paper, K denotes a number field, that is, a finite extension of \mathbb{Q} .

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2. RATIONAL POINTS ON SINGULAR FANO VARIETIES

2.1. Singularities. We first set up terminology on singularities. For details, we refer the reader to [Kol13].

Let X be a normal variety over K . We suppose that X is \mathbb{Q} -Gorenstein, that is, the canonical divisor K_X is \mathbb{Q} -Cartier. A *divisor over X* is a prime divisor on a normal modification Y of X (Y is a normal variety which is proper and birational over X). Here we identify prime divisors on different modifications if they give the same valuation of the function field. For a normal modification $f : Y \rightarrow X$ with exceptional prime divisors E_i , we can uniquely write

$$K_Y = f^*K_X + \sum_i a(E_i) \cdot E_i \quad (a(E_i) \in \mathbb{Q}).$$

The number $a(E_i)$ is independent of the modification and it makes sense to define $a(E)$ for a divisor E over X : we call it the *discrepancy* of E . We call E a *crepant divisor* if $a(E) = 0$. We define the *minimal discrepancy* of X by

$$\text{discrep}(X) := \inf_E a(E),$$

where E runs over all divisors over X . We say that X is *terminal* (resp. *canonical*) or that X has only *terminal* (resp. *canonical*) singularities if

$$\text{discrep}(X) > 0 \quad (\text{resp. } \geq 0).$$

2.2. Heights. Next, we recall the notion of heights of points. For details, we refer the reader to [Pey03, CL10].

Let $\text{Val}(K)$ be the set of valuations (places) of K . For $v \in \text{Val}(K)$, we denote by K_v the corresponding completion of K . If $p \in \text{Val}(\mathbb{Q})$ is such that $v \mid p$ and $|\cdot|_p$ denotes the p -adic norm on \mathbb{Q}_p , then we define the *v -adic norm* on K_v by

$$|a|_v := |N_{K_v/\mathbb{Q}_p}(a)|_p.$$

Let X be a quasi-projective variety over K and \mathcal{L} an invertible sheaf on X . For $v \in \text{Val}(K)$ and $x \in \mathcal{L}(K_v)$, the pullback $x^*\mathcal{L}$ to $\text{Spec } K_v$ is regarded as a one-dimensional K_v -vector space. We say that \mathcal{L} is (*adelically*) *metrized* if \mathcal{L} is endowed with data of v -adic norms

$$\|\cdot\|_v : x^*\mathcal{L} \rightarrow \mathbb{R}_{\geq 0}$$

for all $v \in \text{Val}(K)$ and $x \in \mathcal{L}(K_v)$ which satisfy some conditions (for instance, see [Pey03, CL10]).

Given a metrized invertible sheaf \mathcal{L} on X , the *height* of a K -point $x \in X(K)$ is given by

$$H_{\mathcal{L}}(x) := \prod_{v \in \text{Val}(K)} \|s\|_v^{-1} \in \mathbb{R}_{>0}$$

for any $0 \neq s \in x^*\mathcal{L}$. Given two metrized invertible sheaves \mathcal{L}_1 and \mathcal{L}_2 , then we can naturally metrize the tensor product $\mathcal{L}_1 \otimes \mathcal{L}_2$, and the associated height function satisfies

$$H_{\mathcal{L}_1 \otimes \mathcal{L}_2}(x) = H_{\mathcal{L}_1}(x) \cdot H_{\mathcal{L}_2}(x).$$

We say that a Cartier divisor D on X is *metrized* if $\mathcal{O}_X(D)$ is metrized. For a metrized Cartier divisor D , we define heights by

$$H_D(x) := H_{\mathcal{O}_X(D)}(x).$$

We say that a \mathbb{Q} -Cartier (Weil) divisor D on X is *metrized* if mD is metrized, where m is the least positive integer with mD Cartier. For a metrized \mathbb{Q} -Cartier divisor D , we put

$$H_D(x) := H_{mD}(x)^{1/m}$$

with m as above. We often write $H_D(x)$ simply as $H(x)$, when D is understood.

For a finite field extension L/K , we put $X_L := X \otimes_K L$ and D_L to be the pullback of D to X_L . For $x \in X(L)$, regarding x as an element of $X_L(L)$, we define

$$H_D(x) := H_{D_L}(x).$$

Basic properties of heights are as follows.

Lemma 2.1. *Let L/K be a finite field extension.*

- (1) *For two (metrized) \mathbb{Q} -Cartier divisors D and D' and for $x \in X(L)$, we have $H_{D+D'}(x) = H_D(x)H_{D'}(x)$.*
- (2) *For a morphism $f : Y \rightarrow X$ of K -varieties and $y \in Y(L)$, we have $H_{f^*D}(y) = H_D(f(y))$.*
- (3) *For an extension M/L and for $x \in X(L)$, if x_M denotes the composition $\text{Spec } M \rightarrow \text{Spec } L \xrightarrow{x} X$, then*

$$H_D(x_M) = H_D(x)^{[M:L]}.$$

Proof. All these properties are direct consequences of known properties of heights (for instance, see [CL10]) and the definitions above. \square

2.3. Rational points on singular Fano varieties. We review a generalization of Manin's conjecture by Batyrev and Tschinkel [BT98] and Beukers' observation, which concern the density of rational points on singular Fano varieties.

Definition 2.2. A projective variety over K is called a *terminal* (resp. *canonical*) *Fano variety* if it is normal, \mathbb{Q} -Gorenstein and terminal (resp. canonical), and its anti-canonical divisor $-K_X$ is ample.

Let X be a canonical Fano variety. We suppose that $-K_X$ is metrized and X is given the height function H associated to it. For a subset $U \subset X(K)$, we put

$$N_U(B) := \#\{x \in U \mid H(x) \leq B\}.$$

To state a conjecture on the asymptotic behavior of $N_U(B)$, we need two invariants of X . Firstly we consider the Picard number $\rho(X)$ of X , that is, the rank of the Néron-Severi group. Secondly we put $\text{cd}(X)$ to be the number of crepant divisors over X . Note that these invariants may change after an extension of the base field. The following is a modification of a “conjecture” by Batyrev and Tschinkel [BT98, page 323].

Conjecture 2.3. *If U is a suitable subset of X , then for some positive constant c ,*

$$N_U(B) \sim c \cdot B \cdot (\log B)^{\rho(X) + \text{cd}(X) - 1} \quad (B \rightarrow \infty).$$

- Remark 2.4.* (1) In [BT98], the authors put more specific assumptions on X and U . We note that they did not call it a conjecture. Actually we should regard the conjecture above as rather an optimistic expectation. For this reason, we left the inaccuracy on what U is, and did not mention the necessity of a finite extension of the base field.
- (2) The exponent of B was denoted by $\alpha_{\mathcal{L}}(V)$ in [BT98]. In our situation, it is equal to one, since $\mathcal{L} = \omega_X^{-1}$, although it is not generally an invertible sheaf but only a \mathbb{Q} -invertible sheaf, strictly speaking.
- (3) The exponent of $\log B$ was denoted by $\beta_{\mathcal{L}}(V) - 1$ in [BT98]. In our situation, this number is given as follows: let $f : Y \rightarrow X$ be a resolution of singularities and l the number of the discrepant (= not crepant) prime exceptional divisors of f . Then $\beta_{\mathcal{L}}(V)$ is the Picard number of Y minus l . The Picard number of Y is the Picard number of X plus the number of all exceptional prime divisors over X . Therefore, eventually, $\beta_{\mathcal{L}}(V)$ is the Picard number of X plus the number of crepant divisors over X .

On the constant c in the asymptotic formula in the conjecture, we mention the following observation by Beukers (see [Pey03, page 324]).

Observation. *In some examples where X is defined over the ring of integers \mathcal{O}_K , the constant c contains, as a factor, the Euler product*

$$\prod_{v \in \text{Val}(K)_f} \left(1 - \frac{1}{N(v)}\right)^{\rho(X)} \frac{\#X_{R(v)}(R(v))}{N(v)^{\dim X}},$$

where $\text{Val}(K)_f$ is the set of non-archimedian valuations (finite places), $R(v)$ the residue field of K_v and $N(v)$ the cardinality of $R(v)$.

Perhaps motivated by this, Peyre [Pey95] developed a conjecture on the value of the constant c by means of Tamagawa measures. Batyrev and Tschinkel [BT98] then generalized it further. Although their works seem closely related to ours, we do not pursue it in this paper.

2.4. Projective spaces. Now let us focus on the simplest Fano variety, that is, the projective space. Understanding this case well is crucial in our applications. Fortunately there is a precise result by Schanuel [Sch64].

Let $\mathbb{P}^m = \mathbb{P}_K^m$ be the projective m -space over K . For a suitable metric on $\mathcal{O}_{\mathbb{P}^m}(1)$, we have

$$H_{\mathcal{O}(1)}((x_0 : \cdots : x_n)) = \prod_{v \in \text{Val}(K)} \sup_i \|x_i\|_v$$

(see [CL10]). We give a metric to the anti-canonical divisor $-K_{\mathbb{P}^m}$ induced from the isomorphism $\mathcal{O}(-K_{\mathbb{P}^m}) = \mathcal{O}(m+1)$, and suppose that \mathbb{P}^m is given the height function $H = H_{-K_{\mathbb{P}^m}}$ induced from $-K_{\mathbb{P}^m}$ metrized in this way. Namely H is given by

$$H(x) = \prod_{v \in \text{Val}(K)} \sup_i \|x_i\|_v^{m+1}.$$

Theorem 2.5 ([Sch64]). *We have*

$$N_{\mathbb{P}^m(K)}(B) \sim \delta_{K,m} \cdot B.$$

Here we put

$$(2.1) \quad \delta_{K,m} := \frac{(2^{r_1}(2\pi)^{r_2})^{m+1} \cdot (m+1)^{r_1+r_2-1} \cdot h \cdot R}{\zeta_K(m+1) \cdot w \cdot d^{(m+1)/2}},$$

following the standard notation for invariants of K : r_1 and r_2 are the numbers of real and complex archimedean valuations, w the number of roots of unity, h the class number, d the absolute value of the absolute discriminant, R the regulator and ζ_K the Dedekind zeta function.

We also write d as d_K , to specify the field K . We sometimes consider $\delta_{K,m}$ as an approximation of the power $d_K^{-m/2}$ of the discriminant and use it as a *weight* of K , when counting number fields. This is justified by the following result, a consequence of the Siegel-Brauer theorem [Bra47].

Proposition 2.6. *Fix n and m . When K varies among degree n extensions of \mathbb{Q} , then*

$$\log \delta_{K,m} \sim \log(d_K^{-m/2}) \quad (d_K \rightarrow \infty).$$

In particular, for every positive number ϵ ,

$$d_K^{-m/2-\epsilon} \ll \delta_{K,m} \ll d_K^{-m/2+\epsilon}.$$

Proof. The number

$$\frac{(2^{r_1}(2\pi)^{r_2})^{m+1} (m+1)^{r_1+r_2-1}}{\zeta_K(m+1)w}$$

is bounded from above and below. For instance, concerning the zeta value $\zeta_K(m+1)$, we generally have that for $s > 1$,

$$\zeta(ns) = \prod_p (1 - p^{-ns})^{-1} \leq \zeta_K(s) \leq \prod_p (1 - p^{-1})^{-n} = \zeta(s)^n.$$

Here p runs over the prime numbers and $\zeta(s)$ is the Riemann zeta functions.

As for w , let L be the largest cyclotomic field in K . Putting ϕ as Euler's totient function, we have

$$[L : \mathbb{Q}] = \phi(w) \leq n.$$

The boundedness of w is now followed from the finiteness of $\bigcup_{l \leq n} \phi^{-1}(l)$ (see [Gup81]). The first assertion follows from the Siegel-Brauer theorem [Bra47, Theorem 2]: when the degree of K/\mathbb{Q} is fixed, we have

$$\log(hR) \sim \log \sqrt{d} \quad (d \rightarrow +\infty).$$

For the second assertion, for every $\alpha > 1$, there exists a positive integer n such that for every K with $d_K \geq n$,

$$\alpha \cdot \log d_K^{-m/2} \leq \log \delta_{K,m},$$

and

$$d_K^{-\alpha m/2} \leq \delta_{K,m}.$$

This shows

$$d_K^{-\alpha m/2} \ll \delta_{K,m}.$$

Similarly, for $\beta < 1$,

$$\delta_{K,m} \ll d_K^{-\beta m/2}.$$

We have proved the second assertion. \square

3. THE DENSITY OF NUMBER FIELDS

In this section, we briefly review Malle's conjecture and Bhargava's conjecture on the density of number fields. For details, we refer the reader to [Bel05].

Let n be an integer ≥ 2 and G a transitive subgroup of S_n .

Definition 3.1. A G -field (over K) is a field extension L/K of degree n such that the Galois group $\text{Gal}(\hat{L}/K)$ of the Galois closure \hat{L}/K , acting on the set of K -embeddings $\hat{L} \hookrightarrow \mathbb{C}$, is permutation isomorphic to G . Two G -fields are said to be *isomorphic* if they are isomorphic as K -algebras. The set of isomorphism classes is denoted by $G\text{-fie}(K)$. (We will call a G -field a *small* G -field in later sections to distinguish it from its Galois closure.)

For a positive real number B , we put

$$M_{G,K}(B) := \sharp\{L \in G\text{-fie}(K) \mid |N_{K/\mathbb{Q}}(D_{L/K})| \leq B\},$$

where $D_{L/K}$ is the discriminant of L/K . We are interested in the asymptotic behavior of $M_{G,K}(B)$ as B tends to infinity. To state Malle's conjecture, we define two invariants. We put

$$[n] := \{1, 2, \dots, n\},$$

which has the natural G -action.

Definition 3.2. We define the *index* of $g \in G$ by

$$\text{ind}(g) := n - \sharp\{g\text{-orbits in } [n]\}.$$

We put

$$\text{ind}(G) := \min\{\text{ind}(g) \mid 1 \neq g \in G\}.$$

Denoting the algebraic closure of K by \bar{K} , we consider the natural $\text{Gal}(\bar{K}/K)$ -action on the set $\text{Conj}(G)$ of the conjugacy classes of G (see [Ser77, 12.4]). We denote by $K\text{-Conj}(G)$ the quotient of this action and call elements of $K\text{-Conj}(G)$ *K -conjugacy classes* of G .

Definition 3.3. We put

$$\beta(G, K) := \sharp\{[g] \in K\text{-Conj}(G) \mid \text{ind}(g) = \text{ind}(G)\}.$$

Malle's conjecture is as follows.

Conjecture 3.4 ([Mal02, Mal04]). *We have*

$$M_{G,K}(B) \sim c \cdot B^{1/\text{ind}(G)} (\log B)^{\beta(G,K)-1}.$$

Actually there exists a counterexample to this conjecture [Klü05]. One possible way to rescue the situation would be to ask whether the conjecture holds if one replace K with a sufficiently large finite extension. This is related to replacing the base field by a finite extension in variants of Manin's conjecture (see Remark 2.4).

The index, $\text{ind}(G)$, is equal to one if and only if G contains a transposition. If it is the case, then $\beta(G, K) = 1$ [Mal04, Lemma 2.2]. For instance, for $G = S_n$, the conjecture gives

$$M_{S_n, K}(B) \sim c \cdot B.$$

which is a well known conjecture. Moreover, in this case, Bhargava conjectured the precise value of the constant c . According to [Bel05, Conjecture 6.3] and Bhargava's mass formula for étale extensions of a local field [Bha07], we can write it as follows.

Conjecture 3.5. *We have*

$$M_{S_n, K}(B) \sim c \cdot B,$$

where

$$c = \frac{1}{2} \cdot \text{Res}_{s=1} \zeta_K(s) \cdot \prod_{v \in \text{Val}(K)_f} \left(1 - \frac{1}{N(v)}\right) \left(\sum_{i=0}^{n-1} P(n, n-i) N(v)^{-i}\right)$$

and $P(n, n-i)$ is the number of partitions of n by exactly $n-i$ parts.

4. COMPARISON OF CONSTANTS

In this section, we relate constants appearing in Malle's and Bhargava's conjectures with the geometry of a quotient variety $X_{G,m}$ defined as follows. Let $Y_m := (\mathbb{P}_K^m)^n$, the n -th power of the projective m -space over K . For a transitive subgroup $G \subset S_n$, we put $X_{G,m}$ to be the quotient variety Y_m/G by the natural G -action on Y_m . We write the associated Galois cover as

$$(4.1) \quad \pi_{G,m} : Y_m \rightarrow X_{G,m}.$$

4.1. Indices vs. discrepancies.

Definition 4.1. Let $1 \neq G \subset \text{GL}_n(\mathbb{C})$ be a non-trivial finite subgroup. Diagonalizing each element $g \in G$, we can write

$$h^{-1}gh = \text{diag}(\zeta^{a_1}, \dots, \zeta^{a_n}),$$

where $h \in \text{GL}_n(\mathbb{C})$, $\zeta = \exp(2\pi\sqrt{-1}/l)$ and $0 \leq a_i < l$. We define the *age* of g by

$$\text{age}(g) := \frac{1}{l} \sum_{i=1}^n a_i \in \mathbb{Q}$$

and the *age* of G by

$$\text{age}(G) := \min\{\text{age}(g) \mid 1 \neq g \in G\}.$$

We say that $1 \neq g \in G$ is a *pseudo-reflection* if the fixed point locus $(\mathbb{C}^n)^g$ has codimension one.

This representation-theoretic invariant, age, determines the discrepancy of the associated quotient variety:

Proposition 4.2 ([Yas06]). *Suppose that G has no pseudo-reflection. Then*

$$\text{discrep}(\mathbb{C}^n/G) = \min\{\text{age}(g) \mid 1 \neq g \in G\} - 1.$$

For each n , we regard S_n as a subgroup of $\text{GL}_n(\mathbb{C})$ by the standard permutation representation $S_n \hookrightarrow \text{GL}_n(\mathbb{C})$.

Lemma 4.3 (cf. [WY]). *For $g \in S_n \subset \text{GL}_n(\mathbb{C})$,*

$$2 \cdot \text{age}(g) = \text{ind}(g).$$

Proof. From the additivity of age and index, we may suppose that g is the cyclic permutation, $1 \mapsto 2 \mapsto \dots \mapsto n \mapsto 1$. Then a diagonalization of g is

$$\text{diag}(1, \zeta, \dots, \zeta^{n-1})$$

with $\zeta = \exp(2\pi\sqrt{-1}/n)$. We have

$$\text{age}(g) = \frac{1}{n} \sum_{i=0}^{n-1} i = \frac{n-1}{2} = \frac{\text{ind}(g)}{2}.$$

□

Proposition 4.4. *Suppose that $m \cdot \text{ind}(G) \geq 2$. Then $\pi_{G,m}$ is étale in codimension one and*

$$\text{discrep}(X_{G,m}) = \frac{m \cdot \text{ind}(G)}{2} - 1.$$

In particular, $X_{G,m}$ is canonical.

Proof. If $\text{ind}(G) \geq 2$, then G does not contain a transposition. If $\text{ind}(G) = 1$, then $m \geq 2$. It follows that $\pi_{G,m}$ is étale in codimension one. If $X_{G,m} = \bigcup U_i$ is an open cover, then

$$\text{discrep}(X_{G,m}) = \min_i \text{discrep}(U_i).$$

Moreover $\text{discrep}(X)$ is stable under the extension of base field. Therefore we may replace Y_m with $(\mathbb{C}^m)^n$. Then the proposition follows from Proposition 4.2 and Lemma 4.3. □

4.2. K -conjugacy classes of minimal index vs. crepant divisors. Next we interpret the constant $\beta(G, K)$ in Malle's conjecture in terms of the geometry of $X_{G,m}$.

Proposition 4.5. *If $m \cdot \text{ind}(G) \geq 2$, then $\beta(G, K)$ is equal to the number of divisors E over $X_{G,m}$ with $a(E) = \text{discrep}(X_{G,m})$.*

Proof. The proof is similar to the one of [WY, Theorem 5.4]. We only sketch the outline. Let $X := X_{G,m}$ and $X_L := X \otimes_K L$ for a field extension L/K . Thanks to Lemma 4.3, the McKay correspondence [Bat99] states that for a suitable extension L/K , divisors E over $X_L := X \otimes_K L$ with

$$a(E) = \text{discrep}(X_L) = \text{discrep}(X)$$

correspond to conjugacy classes $[g] \in \text{Conj}(G)$ with $\text{ind}(g) = \text{ind}(G)$. We denote by A the set of such divisors over X_L . As such an extension L/K , we can take, for instance, the cyclotomic extension $K(\zeta_l)/K$, where $l = \#G$ and ζ_l is a primitive root

of unity. The set of divisors E over X with $a(E) = \text{discrep}(X)$ is identified with the $\text{Gal}(K(\zeta_l)/K)$ -quotient of A . The correspondence above of A with a set of conjugacy classes is equivariant with respect to actions of $\text{Gal}(K(\zeta_l)/K)$. We can see this, for instance, by using twisted jets [Yas06]. This proves the proposition. \square

We would like to specialize Conjecture 2.3 to the case where $X = X_{G,m}$. Since X now has Picard number one, the conjecture gives

$$N_U(B) \sim c \cdot B \cdot (\log B)^{\text{cd}(X)}.$$

Suppose now that $m \cdot \text{ind}(G) = 2$. For instance, it is the case when $G = S_n$ and $m = 2$ or when $G = A_n$ and $m = 1$. From the proposition above, Malle's conjecture gives

$$M_{G,K}(B) \sim c \cdot B^{m/2} \cdot (\log B)^{\text{cd}(X)}.$$

4.3. Euler products. Concerning constants c in asymptotic formulas of the form

$$c \cdot B^a \cdot (\log B)^b,$$

we mentioned Beukers' observation on the density of rational points and Bhargava's conjecture on the density of S_n -fields. Euler products in their statements are related as follows.

Suppose $G = S_n$ and $m = 2$. Then $X = X_{S_n,2}$ is nothing but the symmetric product $S^n \mathbb{P}^2$. Let Z be the Hilbert scheme of n points on \mathbb{P}^2 . Then the Hilbert-Chow morphism $Z \rightarrow X$ is a crepant resolution (the pullback of K_X coincides with K_Z), as proved by Beauville [Bea83]. Let $X^\circ := S^n \mathbb{A}^2$, the symmetric product of the affine plane, and Z° the Hilbert scheme of n points on \mathbb{A}^2 . They are open subvarieties of X and Z respectively, and the induced map $Z^\circ \rightarrow X^\circ$ is a crepant resolution, too. In fact, the Hilbert scheme is naturally defined over the ring of integers \mathcal{O}_K . Hence we can take the reduction $Z_{R(v)}^\circ$ for $v \in \text{Val}(K)_f$. For instance, from [CV08, Corollary 3.1], we have

$$\#Z_{R(v)}^\circ(R(v)) = \sum_{i=1}^n P(n, i) N(v)^{n+i} = \sum_{i=0}^{n-1} P(n, n-i) N(v)^{2n-i}.$$

Therefore, the Euler product appearing in Conjecture 3.5 is rewritten as

$$\prod_{v \in \text{Val}(K)_f} \left(1 - \frac{1}{N(v)}\right)^{\rho(X)} \frac{\#Z_{R(v)}^\circ(R(v))}{N(v)^{\dim Z}},$$

having the apparent similarity with the Euler product in Beukers' observation. This coincidence seems more mysterious to the author than the ones for constants a and b . He does not know whether discussions in later sections by height zeta functions help us to clarify anything behind (see Remark 6.15).

Remark 4.6. A relation between Bhargava's mass formula [Bha07] and the Hilbert scheme of points was first observed in [WY].

5. GALOIS COVERS AND CORRESPONDENCES OF POINTS

In this section, we show a correspondence between points of $X_{G,m}$ and points of \mathbb{P}_K^m in a slightly generalized situation.

5.1. General Galois covers.

Definition 5.1. A *large G -field* is a Galois extension L/K endowed with an isomorphism $\text{Gal}(L/K) \cong G$. Two large G -fields L and L' are said to be *isomorphic* if there exists a G -equivariant K -isomorphism $L \cong L'$. The set of isomorphism classes is denoted by $G\text{-Fie}(K)$.

Let G be a finite group, Y a K -variety endowed with a faithful G -action and $X := Y/G$ the quotient variety with the quotient morphism $\pi : Y \rightarrow X$.

Definition 5.2. For $L \in G\text{-Fie}(K)$, an L -point $y \in Y(L)$ is called *G -equivariant* if the morphism $y : \text{Spec } L \rightarrow Y$ is G -equivariant. We denote by $Y(L)^G$ the set of G -equivariant L -points.

Note that $Y(L)^G$ is the fixed point locus of the G -action on $Y(L)$ given by $g \cdot y := g \circ y \circ g^{-1}$. We put

$$Y\{K\} := \bigsqcup_{L \in G\text{-Fie}(K)} \frac{Y(L)^G}{Z(G)},$$

where $Z(G)$ is the center of G and identified with the set of G -equivariant K -automorphisms of L . Let $Y^\circ \subset Y$ and $X^\circ \subset X$ be the largest open subvarieties where π is unramified. We similarly put

$$Y^\circ\{K\} := \bigsqcup_{L \in G\text{-Fie}(K)} \frac{Y^\circ(L)^G}{Z(G)} \subset Y\{K\}.$$

For a G -equivariant point $y : \text{Spec } L \rightarrow Y$, the induced morphism between the G -quotients of the source and target gives a K -point of X , and defines a map

$$\pi_* : Y\{K\} \rightarrow X(K).$$

Definition 5.3. We define the set of *primitive K -points* of X by

$$X(K)^{\text{prim}} := X^\circ(K) \setminus \bigcup_{H \subsetneq G} \text{Im}((Y^\circ/H)(K) \rightarrow X^\circ(K)).$$

Remark 5.4. For a K -variety S , a subset $A \subset S(K)$ is called *thin* if A is contained in the image of $T(K) \rightarrow S(K)$ for a generically finite morphism $T \rightarrow S$ which does not admit a rational section. The set $X(K) \setminus X(K)^{\text{prim}}$ is a thin subset of $X(K)$. Colliot-Thélène conjectured that if S is unirational, then $S(K)$ is not thin (see [Ser08, page 30]). The conjecture implies the folklore conjecture that every finite group is the Galois group of some Galois extension of \mathbb{Q} . His conjecture also shows that if Y is rational, then $X(K)^{\text{prim}}$ is not empty.

Lemma 5.5. A K -point $x \in X(K)$ is primitive if and only if the fiber product

$$\text{Spec } K \times_{x, X, \pi} Y$$

is connected and étale over $\text{Spec } K$.

Proof. It is easy to see that $x \in X^\circ(K)$ if and only if $\text{Spec } K \times_{x, X, \pi} Y$ is étale over $\text{Spec } K$. Suppose that x is not primitive. Then there exists a subgroup $H \subsetneq G$ such that x lifts to a K -point of Y°/H . It follows that $\text{Spec } K \times_{x, X} (Y/H)$ is the disjoint union

of $[G : H]$ copies of $\text{Spec } K$. In particular, it is not connected. Hence $\text{Spec } K \times_{x,X,\pi} Y$ is not connected, either.

Conversely suppose that $\text{Spec } K \times_{x,X,\pi} Y$ is not connected. Let H be the stabilizer of a connected component. Then there is a lift of x in $Y/H(K)$. Hence x is not primitive. \square

Proposition 5.6. *The map π_* induces a bijection*

$$Y^\circ\{K\} \rightarrow X(K)^{\text{prim}}.$$

Proof. Let $(y : \text{Spec } L \rightarrow Y) \in Y^\circ\{K\}$ and $x := \pi_*(y) \in X(K)$. Then there exists a natural G -equivariant K -morphism

$$\text{Spec } L \rightarrow \text{Spec } K \times_{x,X,\pi} Y.$$

Since the source and the target are both étale G -torsors over $\text{Spec } K$, this morphism is an isomorphism. Thus the isomorphism class of y is reconstructed from x . This shows that the map $\pi_*|_{Y^\circ\{K\}}$ is an injection into $X(K)^{\text{prim}}$.

Conversely, we start with an arbitrary primitive point $x \in X(K)^{\text{prim}}$. The last lemma shows that $\text{Spec } K \times_{x,X,\pi} Y$ is isomorphic to $\text{Spec } L$ for some $L \in G\text{-Fie}(K)$. The second projection $\text{Spec } K \times_{x,X,\pi} Y \rightarrow Y$ defines an equivariant point $\text{Spec } L \rightarrow Y^\circ$, which is a lift of x . This shows that $\pi_* : Y^\circ\{K\} \rightarrow X(K)^{\text{prim}}$ is surjective. \square

Proposition 5.7. *Let D be a metrized divisor X . For $y \in Y\{K\}$, we have*

$$H_{\pi^*D}(y)^{1/\#G} = H_D(\pi_*(y)).$$

Proof. From Lemma 2.1, for $y \in Y(L)^G$,

$$\begin{aligned} H_{\pi^*D}(y) &= H_D(\text{Spec } L \xrightarrow{y} \text{Spec } Y \rightarrow \text{Spec } X) \\ &= H_D(\text{Spec } L \rightarrow \text{Spec } K \xrightarrow{\pi_*(y)} \text{Spec } X) \\ &= H_D(\pi_*(y))^{[L:K]}. \end{aligned}$$

This shows the assertion. \square

5.2. Permutation actions of a transitive subgroup. Now suppose that G is a transitive subgroup of S_n , acting on $[n] := \{1, 2, \dots, n\}$. For each $i \in [n]$, we put $G_i \subset G$ to be the stabilizer subgroup of i . Since $G_i, i \in [n]$ are conjugate one another, for each $L \in G\text{-Fie}(K)$, the invariant subfields $L^{G_i}, i \in [n]$ are isomorphic over K one another. There exists a map

$$\phi : G\text{-Fie}(K) \rightarrow G\text{-fie}(K), L \mapsto L^{G_1}.$$

The group of K -automorphisms of a small G -field is isomorphic to the opposite group of the centralizer $C_{S_n}(G)$.

Lemma 5.8. *The map ϕ is a $\frac{\#N_{S_n}(G)\#Z(G)}{\#C_{S_n}(G)\#G}$ -to-one surjection.*

Proof. The map is clearly surjective. Let $\text{Sur}_{\text{cont}}(\text{Gal}(\bar{K}/K), G)$ be the set of continuous surjections of $\text{Gal}(\bar{K}/K)$ to G . The natural map

$$\text{Sur}_{\text{cont}}(\text{Gal}(\bar{K}/K), G) \rightarrow G\text{-Fie}(K)$$

can be identified with the quotient map associated to the G -action on $\text{Sur}_{\text{cont}}(\text{Gal}(\bar{K}/K), G)$ by conjugation. Therefore this map is a $\frac{\#G}{\#Z(G)}$ -to-one surjection. On the other hand, the natural map

$$(5.1) \quad \text{Sur}_{\text{cont}}(\text{Gal}(\bar{K}/K), G) \rightarrow G\text{-fie}(K)$$

is identified with the restriction of the natural map

$$\text{Hom}_{\text{cont}}(\text{Gal}(\bar{K}/K), S_n) \rightarrow n\text{-fie}(K),$$

where $n\text{-fie}(K)$ is the set of isomorphism classes of degree n extensions L/K . The last map is, in turn, identified with the quotient map associated to the S_n -action. Hence map (5.1) is a $\frac{\#N_{S_n}(G)}{\#C_{S_n}(G)}$ -to-one surjection. We have proved the lemma. \square

From the lemma, counting large G -fields and counting small G -fields are equivalent problems.

Let W be a K -variety and let $Y := W^n$, which has a natural G -action. For $L \in G\text{-Fie}(K)$, $y \in Y(L)^G$ and $i \in [n]$, the composition

$$\text{Spec } L \xrightarrow{y} W^n \xrightarrow{p_i} W$$

is stable under the G_i -action on the source, where p_i is the i -th projection. Therefore we obtain a morphism

$$\psi_i(y) : \text{Spec } L^{G_i} \rightarrow W.$$

Lemma 5.9. *For each $i \in [n]$, the induced map*

$$(5.2) \quad \psi_i : Y(L)^G \rightarrow W(L^{G_i}), \quad y \mapsto \psi_i(y)$$

is bijective.

Proof. The given point $y \in Y(L)^G$ is reconstructed from the induced point $w = \psi_i(y) \in W(L^{G_i})$. Indeed, if $\sigma_j \in G$ is any element sending i to j , then

$$(5.3) \quad y = \prod_{i=1}^n (\text{Spec } L \xrightarrow{\sigma_j} \text{Spec } L \xrightarrow{\alpha} \text{Spec } L^{G_i} \xrightarrow{w} W),$$

where α is the morphism corresponding to the inclusion $L^{G_i} \subset L$. This proves the lemma. \square

For $i, j \in [n]$, let

$$\alpha_{ij} : W(L^{G_i}) \rightarrow W(L^{G_j})$$

be the unique map such that $\alpha_{ij} \circ \psi_i = \psi_j$. Any element $g \in G = \text{Gal}(L/K)$ with $g(j) = i$ gives an isomorphism $L^{G_i} \rightarrow L^{G_j}$, being independent of the choice of g . This isomorphism induces the map α_{ij} .

Definition 5.10. With notation as above, we say that a point $x \in W(L^{G_i})$ is *original* if for any nontrivial normal subgroup $1 \neq H \triangleleft G$, $x \notin W(L^{G_i, H})$. Here we identify $W(L^{G_i, H})$ as a subset of $W(L^{G_i})$ in the obvious way. We denote the set of original points in $W(L^{G_i})$ by $W(L^{G_i})^{\text{orig}}$.

Obviously $\alpha_{ij}(W(L^{G_i})^{\text{orig}}) = W(L^{G_j})^{\text{orig}}$. We also note that for $1 \neq H \triangleleft G$ and $i \in [n]$, we have $H \not\subset G_i$ and $L^{\langle G_i, H \rangle} \subsetneq L^{G_i}$. Indeed, if a normal subgroup N of G is contained in some G_i , then, since $G_i, i \in [n]$ are conjugate one another, N is contained in all G_i . Therefore $N = 1$.

Lemma 5.11. *We have*

$$\psi_i(Y^\circ(L)^G) = W(L^{G_i})^{\text{orig}}.$$

Proof. Suppose that $\psi_i(y)$ is not original for some i , and hence for all i . There exists a non-trivial normal subgroup H of G such that for every i , $\psi_i(y) \in W(L^{\langle G_i, H \rangle})$. Let us write $y = \prod_{i=1}^n y_i$ with $y_i \in W(L)$. For $h \in H$ and $i \in [n]$, we have

$$y_{hi} = y_i.$$

Hence $y \in Y^H \subset Y \setminus Y^\circ$.

Conversely, suppose that $y \in Y^g(L)$ for some $1 \neq g \in G$. For each i , $y_i : \text{Spec } L \rightarrow W$ factors through $\text{Spec } L^g$ and $\text{Spec } L^{G_i}$, and hence $\text{Spec } L^g \cap L^{G_i}$. Since, for each $i, j \in [n]$, we have the commutative diagram

$$\begin{array}{ccc} \text{Spec } L^{G_i} & \xrightarrow{\psi_i(x)} & W \\ \cong \downarrow & \nearrow \psi_j(x) & \\ \text{Spec } L^{G_j} & & \end{array},$$

each morphism $\psi_i(x)$ factors also through $\text{Spec } L^h \cap L^{G_i}$ for all elements h conjugate to g . In consequence, $\psi_i(x)$ factors through $\text{Spec } L^{\langle G_i, H \rangle}$ with H the normal closure of $\{g\}$. \square

Definition 5.12. We put

$$\begin{aligned} W[K]_G &:= \bigsqcup_{L \in G\text{-Fie}(K)} \frac{W(L^{G_1})}{Z(G)}, \\ W[K]_G^{\text{orig}} &:= \bigsqcup_{L \in G\text{-Fie}(K)} \frac{W(L^{G_1})^{\text{orig}}}{Z(G)}. \end{aligned}$$

Lemma 5.13. *For each $L \in G\text{-Fie}(K)$, the $Z(G)$ -action on $W(L^{G_1})^{\text{orig}}$ is free.*

Proof. Let $g \in Z(G)$. By definition, any original point $w : \text{Spec } L^{G_1} \rightarrow W$ does not factor through $\text{Spec } L^{\langle G_1, g \rangle}$. Namely, if \bar{w} is the image of the unique point of $\text{Spec } L^{G_1}$ by the morphism w , then the image of

$$w^* : \mathcal{O}_{W, \bar{w}} \rightarrow L^{G_1}$$

is not contained in $L^{\langle G_1, g \rangle}$. This shows $g \cdot w \neq w$. \square

The maps ψ_1 for all $L \in G\text{-Fie}(K)$ induce a map

$$\psi : Y\{K\} \rightarrow W[K]_G.$$

Restricting it, we obtain a bijection, which we denote by the same symbol:

$$\psi : Y^\circ\{K\} \rightarrow W[K]_G^{\text{orig}}.$$

Proposition 5.14. *Let D be a metrized Cartier divisor on W and define a metrized Cartier divisor on Y by*

$$E := \sum_{i=1}^n p_i^* D.$$

For $y \in Y \setminus \{K\}$,

$$H_E(y) = H_D(\psi(y))^{\sharp G}.$$

Proof. Let $\sigma_i \in G$ be such that $\sigma_i(1) = i$. We have the following commutative diagram.

$$\begin{array}{ccc} \mathrm{Spec} L & \xrightarrow{y \circ \sigma_i} & W^n \\ \downarrow & & \downarrow p_i \\ \mathrm{Spec} L^{G_1} & \xrightarrow[\psi(y)]{} & W \end{array}$$

From Lemma 2.1,

$$H_{p_i^* D}(y) = H_{p_i^* D}(y \circ \sigma_i) = H_D(p_i \circ y \circ \sigma_i) = H_D(\psi(y))^{\sharp G_1}.$$

Since $\sharp G = n \cdot \sharp G_1$,

$$H_E(y) = \prod_{i=1}^n H_{p_i^* D}(y) = H_D(\psi(y))^{\sharp G},$$

as desired. \square

For such a divisor E as in the lemma, we define a \mathbb{Q} -Cartier divisor \bar{E} on X as $\frac{1}{\sharp G}(\pi_* E)$ and metrize it so that $\pi^* \bar{E} = E$ as metrized divisors.

Remark 5.15. Indeed we can always metrize \bar{E} in this way. The divisor E corresponds to a G -invertible sheaf \mathcal{L} on Y . Hence the divisor $\sharp G \cdot \bar{E} = \pi_* E$ is a Cartier divisor corresponding to the G -invariant part of $\pi_*(\mathcal{L}^{\otimes \sharp G})$, which naturally inherits the metric on $\mathcal{L}^{\otimes \sharp G}$. This defines a metric on \bar{E} satisfying the desired property.

Consider the bijection

$$(5.4) \quad \tau := \psi \circ (\pi_*)^{-1} : X(K)^{\mathrm{prim}} \rightarrow W[K]_G^{\mathrm{orig}}.$$

From Propositions 5.7 and 5.14, we obtain the following corollary.

Corollary 5.16. *For $x \in X(K)^{\mathrm{prim}}$,*

$$H_{\bar{E}}(x) = H_D(\tau(x)).$$

6. HEIGHT ZETA FUNCTIONS AND POSSIBLE APPLICATIONS

6.1. Height zeta functions. The height zeta function is the main analytic tool in studies of the density of rational points, and also in our study of relating it with the density of number fields. For details of height zeta functions, we refer the reader to [CL10].

Definition 6.1. Let S be a K -variety endowed with a height function H . For a certain set U of points of S , we define the *height zeta function* of U to be

$$Z_U(s) := \sum_{x \in U} H(x)^{-s},$$

for $s \in \mathbb{C}$ whenever the series absolutely converges.

Let $N_U(B) := \#\{x \in U \mid H(x) \leq B\}$. If $a \in \mathbb{R}$ is the abscissa of absolute convergence of $Z_U(s)$ and if $Z_U(s)$ has a meromorphic continuation to a neighborhood of $s = a$, then a version of Tauberian theorem implies that

$$N_U(B) \sim c \cdot B^a \cdot (\log B)^{b-1},$$

where b is the order of the pole of $Z_U(s)$ at $s = a$ and

$$c = a \cdot (b-1)! \cdot \lim_{s \rightarrow a} (s-a)^b \cdot Z_U(s).$$

From now on, we suppose that projective spaces \mathbb{P}^m have anti-canonical divisors metrized as in Section 2.4 and the associated height functions. We need the following precise description of the height zeta function of a projective space.

Theorem 6.2 ([FMT89]). *Let L be a number field. The height zeta function $Z_{\mathbb{P}^m(L)}(s)$ absolutely converges for $\Re(s) > 1$ and has a meromorphic continuation to the whole s -plane. Moreover it has a simple pole at $s = 1$ whose residue is equal to $\delta_{L,m}$ defined in (2.1).*

In what follows, we suppose that G is a transitive subgroup of S_n and $m \cdot \text{ind}(G) \geq 2$. We suppose that the quotient variety $X := (\mathbb{P}^m)^n / G$ is given the metrized anti-canonical divisor such that we have an equality of metrized divisors,

$$\pi^*(-K_X) = \sum p_i^*(-K_{\mathbb{P}^m}),$$

where $\pi : (\mathbb{P}^m)^n \rightarrow X$ is the quotient map and $p_i : (\mathbb{P}^m)^n \rightarrow \mathbb{P}^m$ is the i -th projection. From Lemma 5.13 and Corollary 5.16, we obtain the following result.

Theorem 6.3. *We have*

$$Z_{X(K)^{\text{prim}}}(s) = \frac{1}{\#Z(G)} \sum_{L \in G\text{-Fie}(K)} Z_{\mathbb{P}^m(L^{G_1})^{\text{orig}}}(s),$$

whenever the series $Z_{X(K)^{\text{prim}}}(s)$ absolutely converges.

The equality can be considered as a *mass formula* of (large or small) G -fields, a formula for counting G -fields with weight $\frac{1}{\#Z(G)} \cdot Z_{\mathbb{P}^m(L^{G_1})^{\text{orig}}}(s)$. This is also a global analogue of the wild McKay correspondence [WY, Yas14, Yasa, Yasb] in the case of local fields.

Remark 6.4. In the situation of Section 5.1, we similarly have

$$Z_{X(K)^{\text{prim}}}(s) = \frac{1}{\#Z(G)} \cdot \sum_{L \in G\text{-Fie}(K)} Z_{Y^\circ(L)^G}(s).$$

Using this equality instead of the one above, we might be able to apply arguments below to counting of G -fields with respect to various weights other than powers of discriminants, as proposed in [EV05, Section 4.2]. For this purpose, we need to know the height zeta function $Z_{Y^\circ(L)^G}(s)$ as well as we know now the usual height zeta function of a projective space.

Now we show that function $Z_{\mathbb{P}^m(L^{G_1})^{\text{orig}}}(s)$ has as nice properties as $Z_{\mathbb{P}^m(L^{G_1})}(s)$ does.

Definition 6.5. Define the following set of subgroups of G

$$A(G) := \{\langle G_1, H \rangle \mid H \triangleleft G\},$$

For $I \in A(G)$, we call a point $x \in W(L^I)$ *original* if x is not in the image of $W(L^J)$ for any $J \in A(G)$ with $I \subsetneq J$. We denote by $W(L^I)^{\text{orig}}$ the set of original L^I -points.

For $I = G_1$, this definition of original coincides with Definition 5.10. Obviously we have

$$\mathbb{P}^m(L^{G_1}) = \bigsqcup_{I \in A(G)} \overline{\mathbb{P}^m(L^I)^{\text{orig}}},$$

where $\overline{\mathbb{P}^m(L^I)^{\text{orig}}}$ is the image of $\mathbb{P}^m(L^I)^{\text{orig}}$ by the natural injection $\mathbb{P}^m(L^I)^{\text{orig}} \rightarrow \mathbb{P}^m(L^{G_1})$. From Lemma 2.1,

$$Z_{\mathbb{P}^m(L^{G_1})}(s) = \sum_{I \in A(G)} Z_{\mathbb{P}^m(L^I)^{\text{orig}}}([I : G_1] \cdot s).$$

Similarly, for each $I \in A(G)$,

$$Z_{\mathbb{P}^m(L^I)}([I : G_1] \cdot s) = \sum_{\substack{J \in A(G) \\ I \subsetneq J}} Z_{\mathbb{P}^m(L^J)^{\text{orig}}}([J : G_1] \cdot s).$$

Proposition 6.6. *We have*

$$\begin{aligned} Z_{\mathbb{P}^m(L^{G_1})^{\text{orig}}}(s) &= \sum_{I \in A(G)} \mu(I) \cdot Z_{\mathbb{P}^m(L^I)}([I : G_1] \cdot s) \\ &= Z_{\mathbb{P}^m(L^{G_1})}(s) + \sum_{\substack{I \in A(G) \\ I \neq G_1}} \mu(I) \cdot Z_{\mathbb{P}^m(L^I)}([I : G_1] \cdot s), \end{aligned}$$

where $\mu(I) = \mu(G_1, I) \in \mathbb{Z}$ is the Möbius function for the poset $A(G)$.

Proof. This follows from the Möbius inversion formula (see [Sta97, Proposition 3.7.2]). \square

The following is a direct consequence of the proposition.

Corollary 6.7. *For $L \in G\text{-Fie}(K)$, the function $Z_{\mathbb{P}^m(L^{G_1})^{\text{orig}}}(s)$ absolutely converges for $\Re(s) > 1$, admits a meromorphic continuation to the whole s -plane, and has a simple pole at $s = 1$ with residue $\delta_{L^{G_1}, m}$.*

Example 6.8. For $G = S_n$, we have $A(G) = \{G_1, S_n\}$. Hence

$$\mathbb{P}^m(L^{G_1})^{\text{orig}} = \mathbb{P}^m(L^{G_1}) \sqcup \overline{\mathbb{P}^m(K)},$$

and

$$Z_{\mathbb{P}^m(L^{G_1})^{\text{orig}}}(s) = Z_{\mathbb{P}^m(L^{G_1})}(s) - Z_{\mathbb{P}^m(K)}(ns).$$

6.2. The density of primitive points. We continue to work in the following situation: $G \subset S_n$ is a transitive subgroup, $X = (\mathbb{P}^m)^n/G$ and $m \cdot \text{ind}(G) \geq 2$. Let us consider the following more specific version of Conjecture 2.3.

Conjecture 6.9. *There exists a positive constant c such that*

$$N_{X(K)^{\text{prim}}}(B) \sim c \cdot B \cdot (\log B)^{\text{cd}(X)} \quad (B \rightarrow \infty).$$

Remark 6.10. The set of primitive points $X(K)^{\text{prim}}$ is obtained by removing a thin subset from $X(K)$ (Remark 5.4). Removing a thin subset in the context of Manin's conjecture is suggested already in [Pey03, page 345]. The whole set $X(K)$ would not satisfy the same asymptotic formula as above. For instance, if a subgroup $H \subset S$ is not transitive, then $(\mathbb{P}^m)^n/H$ has Picard number ≥ 2 and the images of its K -points on $(\mathbb{P}^m)^n/S_n$ are expected to outnumber the primitive K -points. This would be related to the phenomenon that étale extensions outnumber field extensions (for instance, see [Bha05, pages 1034-1035]). For, if we extend bijection (5.4), general K -points of X correspond to E -points of W for an étale K -algebra E of degree n .

We would like to show partial implications between Conjecture 6.9 and Malle's conjecture, by using the equality above of height zeta functions and assuming that the following auxiliary conjectures hold.

Conjecture 6.11. *The height zeta function $Z_{X(K)^{\text{prim}}}(s)$ absolutely converges for $\Re(s) > 1$, admits a meromorphic continuation to a neighborhood of $s = 1$ with a pole at $s = 1$.*

The first part of this conjecture follows from Conjecture 6.9. Although assuming Conjecture 6.11 might not be really necessary, it would simplify arguments.

Although we might encounter an unwished issue of limits, it seems reasonable to expect the following.

Conjecture 6.12. *The function $Z_{X(K)^{\text{prim}}}(s)$ has a simple pole at $s = 1$ if and only if*

$$\sum_{F \in G\text{-fie}(K)} \delta_{F,m} < \infty.$$

If these equivalent conditions hold, then

$$\text{Res}_{s=1} Z_{X(K)^{\text{prim}}}(s) = \frac{\#N_{S_n}(G)}{\#C_{S_n}(G) \cdot \#G} \cdot \sum_{F \in G\text{-fie}(K)} \delta_{F,m}.$$

Theorem 6.13. *Let $G \subset S_n$ be a transitive subgroup, m a positive integer with $m \cdot \text{ind}(G) > 2$, and $X := (\mathbb{P}^m)^n/G$. Suppose that for these G and X , Conjectures 6.11, 6.12 and 3.4 hold. Then, Conjecture 6.9 holds for X .*

Proof. Since $m \cdot \text{ind}(G) > 2$, X is terminal, and $\text{cd}(X) = 0$. Therefore it suffices to show that $Z_{X(K)^{\text{prim}}}(s)$ has a simple pole at $s = 1$. From Conjecture 6.12, this is equivalent to $\sum_{F \in G\text{-fie}(K)} \delta_{F,m} < \infty$. Conjecture 3.4 shows

$$\limsup_{B \rightarrow +\infty} \frac{\log M_{G,K}(B)}{\log B} = \frac{1}{\text{ind}(G)}.$$

From the lemma below, the Dirichlet series

$$(6.1) \quad \sum_{F \in G\text{-fie}(K)} |N_{K/\mathbb{Q}}(D_{F/K})|^{-s} = d_K^{-ns} \cdot \sum_{F \in G\text{-fie}(K)} d_F^{-s}$$

has $1/\text{ind}(G)$ as the abscissa of convergence. In particular, the series converges for $s = m/2$. From Proposition 2.6, $\sum_{F \in G\text{-fie}(K)} \delta_{F,m} < \infty$, as desired. \square

Lemma 6.14 ([Man69, T. I.2.7]). *For a Dirichlet series*

$$f(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^{-s}},$$

put

$$a := \limsup_{n \rightarrow +\infty} \frac{\log(\sum_{m \leq n} a_m)}{\log n}.$$

If $0 < a < \infty$, then a is the abscissa of convergence of $f(s)$.

Malle's conjecture is known to hold for abelian groups, for instance. On the other hand, even for abelian groups, the quotient variety $X = (\mathbb{P}^m)^n/G$ is not generally rational. Thus, once Conjectures 6.11 and 6.12 are verified, Theorem 6.13 would give non-trivial affirmative results.

Remark 6.15. It appears harder, but more interesting to consider the case where $m \cdot \text{ind}(G) = 2$, (equivalently, $X = (\mathbb{P}^m)^n/G$ is canonical but not terminal). In this case, Conjectures in this paper suggest that, when taking the limit

$$\lim_{N \rightarrow +\infty} \sum_{\substack{L \in G\text{-Fie}(K) \\ d_{L/G_1} \leq N}} Z_{\mathbb{P}^m(L^{G_1})^{\text{orig}}}(s),$$

the order of pole at $s = 1$ would jump up from one to

$$\text{cd}(X) + 1 = \beta(G, K) + 1.$$

However the author does not know either how to show this from information on the density of G -fields, or how to get a positive result on Malle's conjecture from information on $Z_{X(K)^{\text{prim}}}(s)$.

The condition $m \cdot \text{ind}(G) = 2$ is restrictive, although it is satisfied by the important case where $G = S_n$ and $m = 2$. It might be an interesting problem to ask what is a substitute for $(\mathbb{P}^m)^n/G$ when m is only a rational number.

6.3. The density of number fields. For general G and n , the best known upper bounds for $M_{G,K}(B)$ are the one by Schmidt [Sch95],

$$M_{G,K}(B) \ll B^{\frac{n+2}{4}},$$

and the one by Ellenberg and Venkatesh [EV06],

$$N_{G,K}(B) \ll (B \cdot C_1)^{\exp(C_2 \sqrt{\log n})},$$

with C_1 and C_2 positive constants. For $G = A_n$, Larson and Rolen [LR13] showed

$$N_{A_n,K}(B) \ll B^{\frac{n^2-2}{4(n-1)}} \cdot (\log B)^{2n+1}.$$

In particular, a bound independent of n has not been obtained yet.

Theorem 6.16. *Suppose that Conjectures 6.9, 6.11 and 6.12 hold for the given transitive subgroup G and $m = \lceil 3/\text{ind}(G) \rceil$. Then, for any $\epsilon > 0$,*

$$M_{G,K}(B) \ll B^{\frac{1}{2} \lceil \frac{3}{\text{ind}(G)} \rceil + \epsilon}.$$

Proof. We first note that from known bounds above,

$$0 < \limsup_{B \rightarrow +\infty} \frac{\log M_{G,K}(B)}{\log B} < \infty.$$

Therefore we can apply Lemma 6.14 to Dirichlet series (6.1). For $m = \lceil 3/\text{ind}(G) \rceil$, since $m \cdot \text{ind}(G) > 2$, $X = (\mathbb{P}^m)^n/G$ is a terminal Fano variety. From Conjectures 6.9 and 6.12, we have

$$\sum_{F \in G\text{-fie}(K)} \delta_{F,m} < \infty.$$

From Proposition 2.6, for any $\epsilon > 0$,

$$\sum_{F \in G\text{-fie}(K)} |N_{K/\mathbb{Q}}(D_{F/K})|^{-m/2-\epsilon} = d_K^{-n(m/2+\epsilon)} \cdot \sum_{F \in G\text{-fie}(K)} d_F^{-m/2-\epsilon} < \infty.$$

Hence the abscissa of convergence of the Dirichlet series is at most $m/2$. From Lemma 6.14,

$$\limsup_{B \rightarrow +\infty} \frac{\log M_{G,K}(B)}{\log B} \leq \frac{m}{2}.$$

For every $\epsilon > 0$, there exists B_0 such that for every $B \geq B_0$,

$$\frac{\log M_{G,K}(B)}{\log B} \leq \frac{m}{2} + \epsilon,$$

equivalently

$$M_{G,K}(B) \leq B^{m/2+\epsilon}.$$

This shows

$$M_{G,K}(B) \ll B^{m/2+\epsilon}.$$

□

When $G = S_n$, the theorem gives

$$M_{S_n, K}(B) \ll B^{3/2+\epsilon},$$

and when $G = A_n$,

$$M_{S_n, K}(B) \ll B^{1+\epsilon}.$$

Thus, once Conjectures 6.9, 6.12 and 6.11 are proved, we would considerably improve present bounds for general n , although these conjectures might be equally hard to prove.

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